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LETTER TO THE EDITOR

A new RG approach to site percolation in two dimensions

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Abstract. A renormalisation group transformation is developed for two-dimensional site-percolation problems by using a scaling transformation in real space. A transition matrix A is defined for each cell, and the renormalised probability $p'(p)$ of occupation of the cell is identified with the dominant eigenvalue $R_1(p)$ of the transition matrix. A simple RG transformation has been applied on the square lattice up to cells of size 6×6 and the results for critical probability p_c and exponent ν are given. A modified RG has been applied to the three planar lattices, and p_c and ν have been calculated for the simplest choice of the cells. The modified RG transformation seems to yield better results as the coordination number of the lattice increases.

In recent years, there has been a tremendous amount of research activity related to percolation problems (Essam 1972, Shante and Kirkpatrick 1971). Most interesting among these problems are the calculations of the critical exponents and the critical probability. We will attempt here to develop a new RG approach and estimate the critical probability p_c and the critical exponent ν. The approach is to use a scaling procedure in a real space (Young and Stinchcombe 1975, Stinchcombe and Watson 1976, Kirkpatrick 1977, Reynolds *et al* 1977, 1978, Sarychev 1977, Yuge and Murase 1978) based on the original ideas of Niemeijer and van Leeuwen (1976) but developed in the context of the percolation problem by Stinchcombe and co-workers. The point of view of the above work is that the principal effect of rescaling length is to modify p, the probability of occupation, and that the critical probability can be identified as a fixed point of this transformation of p. We take the same point of view and develop a method to determine this transformation law of p. This transformed probability $p'(p)$ is some 'effective' probability of occupation of the cell. The definition of $p'(p)$ is arrived at by considering connections that ensure the occurrence of infinite clusters of occupied sites. Our approach is to define a transition matrix $A(p)$, and $p'(p)$ is identified with the dominant eigenvalue $R_1(p)$ of $A(p)$. This eigenvalue $R_1(p)$ (and therefore $p'(p)$) has the property that it lies between 0 and 1, and that as the cell size increases, it tends to zero for $p<p_c$ and rises sharply at p just above p_c to unity (figure 2).

For the sake of clarity, we consider a simple square lattice, but the theory is applicable to any lattice. Consider a 2×2 cell C and the nearest-neighbour sites t_1 and t_2 of the neighbouring cell C_1 (figure 1(*a*)). Let $|i\rangle$ and $|j\rangle$ denote any of the four possible states of t_1 and t_2 and s_1 and s_2 respectively, and define the transition matrix $A(p)$ with elements $A_{ij}\equiv\langle j|A|i\rangle$ as follows. A_{ij} is the conditional probability that given a state $|i\rangle$ of t_1 and t_2, there are connecting paths of occupied sites between state $|i\rangle$ and state $|j\rangle$ (of

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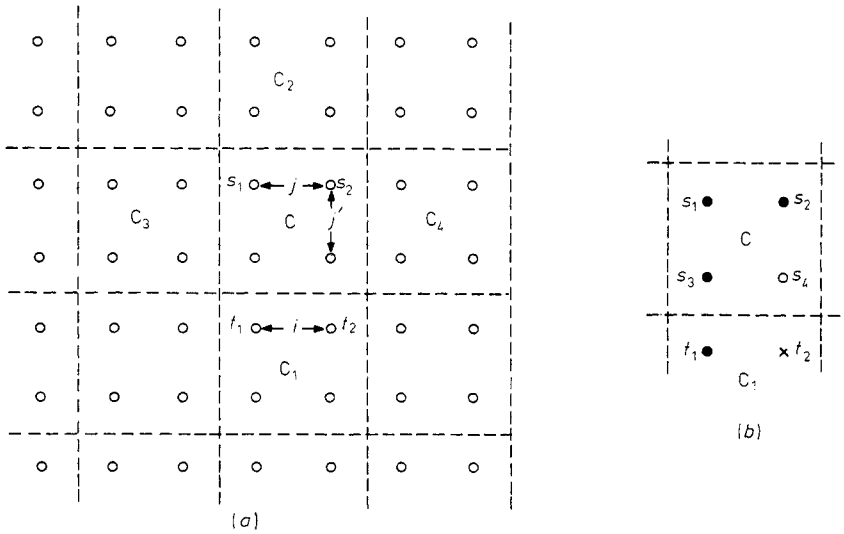

Figure 1. Simple RG on a square lattice.

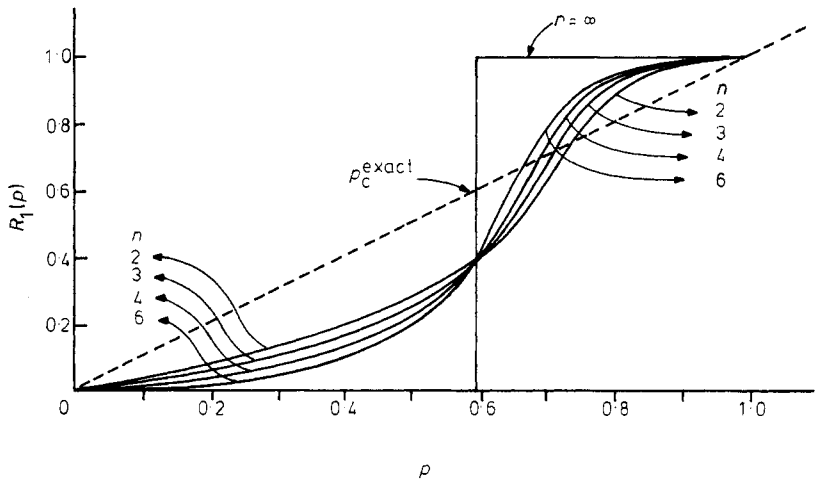

Figure 2. $R_1(p)$ and the behaviour of the fixed point (simple RG).

s_1 and s_2). For example, consider figure 1(b) where the 'in state' $|i\rangle$ is $|\bullet\,\times\rangle$ (the full circle $\bullet$ denotes an occupied site with probability p and the cross $\times$ denotes an unoccupied site with probability $q = 1-p$) and the 'out state' is $|\bullet\,\bullet\rangle$. For this case, $\langle\bullet\,\bullet|A|\bullet\,\times\rangle = p^3 =$ probability that s_1, s_2 and s_3 are all occupied.

The matrix $A(p)$ is given by

$$A(p) = \begin{pmatrix} p^2 s & p^2 q & p^2 q & 0 \\ p^3 & p^2 q & p^3 q & 0 \\ p^3 & p^3 q & p^2 q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad s = p^2 + 2pq$$

where the states $|\bullet\bullet\rangle$, $|\bullet\times\rangle$, $|\times\bullet\rangle$ and $|\times\times\rangle$ are numbered from 1 to 4. Let us introduce a periodic boundary condition along the vertical direction on the $\sqrt{N}\times\sqrt{N}$ square lattice. Then $P_v = \mathrm{Tr}(A^{\sqrt{N}})$ $(N\to\infty)$ denotes the probability of having connected paths of occupied sites along any vertical strip of cells $C_1, C, C_2, \ldots$, etc. If P is the total probability that there are such paths on each strip, we have

$$P = \mathrm{Tr}(A^N) = (R_1(p))^N \qquad N\to\infty \tag{1}$$

where $R_1(p)$ is the dominant eigenvalue of $A(p)$. It is evident that $A(p)$ is a non-negative matrix and, therefore, according to the Perron–Frobenius theorem has a *real* and *positive* dominant eigenvalue (Gantmacher 1959). Now it is natural to identify $R_1(p)$ as the 'effective probability' of occupation $p'(p)$ of the cell:

$$p'(p) = R_1(p). \tag{2}$$

The fixed point p^* is given by

$$p'(p^*) = p^* \tag{3}$$

and the critical index ν (Wilson and Kogut 1974) by

$$\nu = \ln b / R_1'(p^*) \tag{4}$$

where b is the length scale change and $R_1'(p) = \partial R_1(p)/\partial p$.

The results for p_n^* and ν_n for $n = 2, 3, 4, 5$ and 6 ($n\times n$ is the size of the cell under transformation) are given in table 1. We note that the convergence of p_n^* towards $p_c^{(\mathrm{series})} = 0{\cdot}593 \pm 0{\cdot}002$ (Sykes and Glen 1976, Sykes *et al* 1976a, b, c) is very poor. It is believed that ν for $d = 2$ percolation problems is around 1·3 and therefore the convergence of ν_n towards ν is also very slow.

Table 1. Unidirectional RG on a square lattice.

Cell size	p^*	$R_1'(p^*)$	ν
2×2	0·734	1·560	1·559
3×3	0·706	2·038	1·543
4×4	0·688	2·480	1·526
5×5	0·676	2·889	1·517
6×6	0·667	3·284	1·507

In order to get better estimates of p_c and ν, we consider the in states $|i\rangle$ determined by the four sites t_1, t_2, t_3 and t_4 (figure 3(a)). There are 16 such states. The out states $|j\rangle$ are determined by the three sites s_1, s_2 and s_3. We 'mentally' split the middle site s_2 into two sites s_1' and s_3' (figure 3(b)) and define

$$A_{ij} = 0 \qquad \text{if } j \in S \tag{5}$$

where S is the set of states of the four sites s_1, s_1', s_3' and s_3 in which s_1' and s_3' are not in the same state. With $|j\rangle$ now as any of the 16 states of s_1, s_1', s_3' and s_3 and with the restriction (5) on A, we define the matrix element A_{ij} as follows. A_{ij} is the conditional probability that given an in state $|i\rangle = |i_1\rangle\otimes|i_2\rangle$, there are connecting paths of occupied sites from either $|i_1\rangle$ or $|i_2\rangle$ or both to both $|j_1\rangle$ and $|j_2\rangle$ ($|j\rangle = |j_1\rangle\otimes|j_2\rangle$). This definition of $A(p)$ ensures that if such connections are allowed for each cell, then each cell is a part of

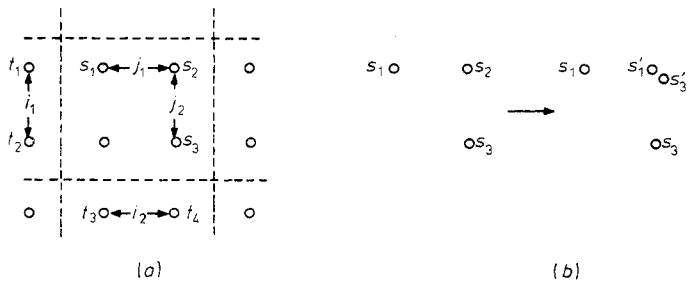

Figure 3. Modified RG on a square lattice: (*a*) in and out states; (*b*) splitting of the three out sites.

some infinite cluster of occupied sites. To see this, we pick any arbitrary 'occupied' cell. We must have come into this cell through at least one of the two faces, one on the left and the other one at the bottom. If we follow this connection, we come to (at least one) another 'occupied' cell which in turn must lead to (at least one) another cell and so on. This implies the occurrence of an infinite cluster. (An 'occupied' cell is one which has connecting paths from $|i\rangle$ to $|j\rangle$.)

We will again identify $p'(p)$, the 'effective' probability of occupation of a cell, with the dominant eigenvalue $R_1(p)$ of $A(p)$ (see equation (2)). Now, $R_1(p)$ is also the dominant eigenvalue of the truncated matrix $\bar{A}(p)$, obtained by limiting $|i\rangle$ and $|j\rangle$ to the following states of $|s_1, s_2, s_3\rangle$: $|\bullet\bullet\bullet\rangle$, $|\bullet\bullet\times\rangle$, $|\times\bullet\bullet\rangle$, $|\bullet\times\bullet\rangle$, numbered from 1 to 4. (The matrix A has the block form

$$A = \left(\begin{array}{c|c} \bar{A} & 0 \\ \hline A' & 0 \end{array}\right)$$

and $\bar{A}$ is the truncated square matrix defined above.) The transition matrix $\bar{A}(p)$ is given by

$$\bar{A}(p) = \begin{pmatrix} p^3 & p^2q & p^2q & p^2q \\ p^3 & p^2q & p^2q & p^2q \\ p^3 & p^2q & p^2q & p^2q \\ p^4 & p^3q & p^3q & p^3q \end{pmatrix}$$

which are identical to the results obtained by Yuge and Murase (1978), who used a different approach.

We will now apply the above modified RG transformation on the hexagonal lattice. We take the four sites s_1, s_2, s_3 and s_4 to provide us with the basic cell on this lattice (see figure 4(*a*)). We will speak of these cells as 'trishools' (trishool is a Hindi word meaning three-pronged figures). The six sites t_1, t_2, t_3, t_4, t_5, t_6 determine the in states. We split each of the four sites s_1, s_2, s_3 and s_4 into two 'half-sites' such that s_1', s_2, s_3, s_3', s_2' and s_4' determine the out states (figure 4(*b*)). We again set the matrix element A_{ij} equal to zero if $|j\rangle$ is an out state in which the states of s_2 and s_2' and s_3 and s_3' are not identical, respectively. The transition matrix $A(p)$ from t sites to s sites can be obtained by the product of two matrices B and C ($A = BC$), where B is the transition matrix from the in states of the six t sites to the out states of the two sites u_1 and u_2 and C is the transition matrix between the u sites and the s sites. We again have to consider the truncated matrix $\bar{A}(p)$ by considering only those six out states of the s sites that contribute

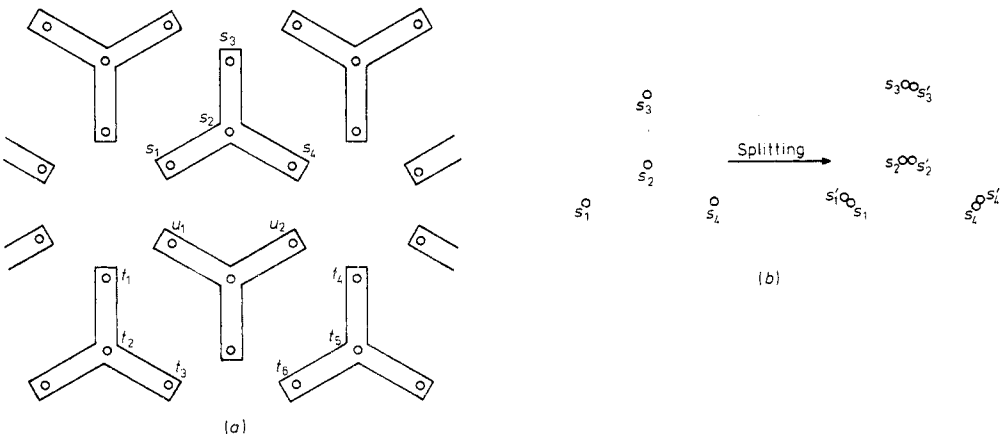

Figure 4. 'Trishool' transformation on a hexagonal lattice: (*a*) in and out states; (*b*) splitting of the four out sites.

non-zero matrix elements C_{ij} and the three states of the u sites that contribute non-zero B_{ij}. The six states of (s_1, s_2, s_3, s_4) are $|\bullet\bullet\bullet\bullet\rangle$, $|\bullet\bullet\bullet\times\rangle$, $|\times\bullet\bullet\bullet\rangle$, $|\bullet\bullet\times\bullet\rangle$, $|\bullet\times\bullet\bullet\rangle$ and $|\bullet\times\times\bullet\rangle$. The three states of $|u_1, u_2\rangle$ are $|\bullet\bullet\rangle$, $|\bullet\times\rangle$ and $|\times\bullet\rangle$. The matrices B and C are given by

$$B(p)=\begin{pmatrix} p^2 & pq & pq \\ p^2 & pq & pq \\ p^2 & pq & pq \\ p^4 & p^3q & p^3q \\ p^4 & p^3q & p^3q \\ p^4 & p^3q & p^3q \end{pmatrix}$$

$$C(p)=\begin{pmatrix} p^4 & p^3q & p^3q & p^3q & p^3q & p^2q^2 \\ p^4 & p^3q & 0 & p^3q & p^3q & p^2q^2 \\ p^4 & 0 & p^3q & p^3q & p^3q & p^2q^2 \end{pmatrix}$$

and

$$A=BC.$$

We find from these matrices that

$$p^*=0{\cdot}721 \qquad \nu=1{\cdot}630.$$

The value of p_c from series calculations (Essam 1972) is

$$p_c=0{\cdot}70\pm0{\cdot}01$$

and our result for p_c is not very unsatisfactory at the lowest size of the cell transformation.

As the final example of the applicability of our RG transformation, we consider the triangular lattice (figure 5(*a*)). The set of t sites t_1, t_2, t_3, t_4, t_5 and t_6 define the in states and the split sites (figure 5(*b*)) s_1, s_1', s_2, s_2' and s_3, s_3' define the out states. The transition matrix elements A_{ij} are non-zero only for those out states that allow connecting paths through all the three faces defined by (s_1, s_2), (s_2, s_3') and (s_3, s_1'). This implies that at

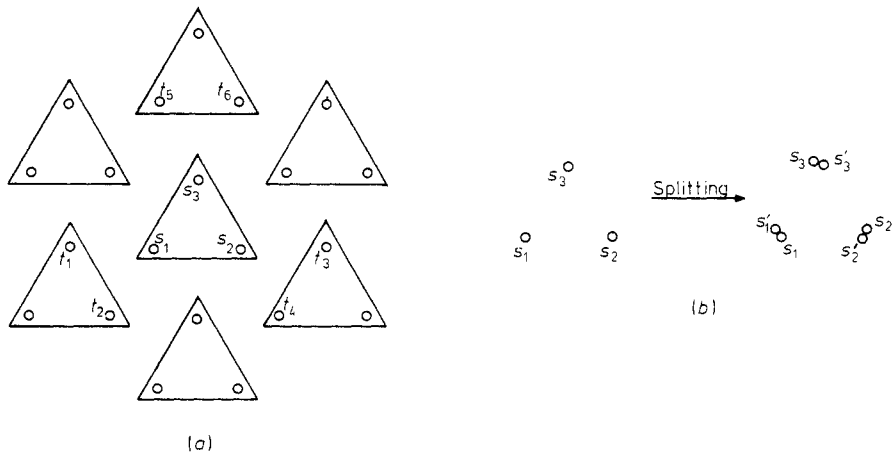

Figure 5. Triangular cell transformation: (*a*) in and out states; (*b*) splitting of the three out sites.

least two of the three sites s_1, s_2 and s_3 must be occupied. The truncated matrix $\bar{A}$ is determined by the following four out states of $|s_1, s_2, s_3\rangle$: $|\bullet\bullet\bullet\rangle$, $|\bullet\times\bullet\rangle$, $|\bullet\bullet\times\rangle$ and $|\times\bullet\bullet\rangle$. It is easily seen that each row of $\bar{A}$ has the same sum, namely p^3+3p^2q. Thus,

$$R_1(p)=p^3+3p^2q$$

(see also Reynolds *et al* 1977) with

$$p^*=\tfrac{1}{2}$$

($p_c=\frac{1}{2}$ exactly for the triangular lattice) and

$$\nu=\ln\sqrt{3}/\ln(\tfrac{3}{2})=1{\cdot}3548.$$

Recent calculations for ν have yielded the following values:

$$\nu=1{\cdot}35\pm0{\cdot}02 \qquad \text{(Gaunt and Sykes 1976)}$$

and

$$\nu=1{\cdot}356\pm0{\cdot}015 \qquad \text{(Reynolds } et\ al \text{ 1978)}$$

and our value for ν for the triangular lattice is in close agreement with the above calculations. Incidently, our results for the triangular lattice are exactly the same as those obtained by Reynolds *et al* (1977), even though the approaches are completely different.

It must be evident that the results of our RG calculations improve as the coordination number of the lattice increases. They are worst for the hexagonal lattice and best for the triangular lattice.

Finally, we conclude that our modified RG transformations have yielded encouraging results considering only the simplest possible choices of the cells have been made. Our transformations are not only more pictorial and closer to the actual physics, that is, the occurrence of infinite clusters, but also give a 'proper' meaning to the concept of effective probability of occupation of a cell. The ideas developed here can be applied to the bond problems as well, and also to three-dimensional lattices.

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